Jeremiah Heller.
Motivic Segal conjecture.
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Joint with Gregersen - Kylling - Rognes - Østvaer.

I. Classical story.

Atiyah: $[BG, \mathbb{Z}\times BU] \xleftarrow{\sim} \mathrm{Rep}(G)^{\wedge}_{I}$.
G finite complex rep. ring

I is the kernel of
the rank map $\mathrm{Rep}(G) \to \mathbb{Z}$.

What about $[BG, \mathbb{Z}\times B\Sigma_\infty]$ or, motivated by Barratt-Priddy-Quillen,
$[BG, QS^0]$,

since $QS^0 \simeq \Omega^\infty\Sigma^\infty S^0$. We get maps from finite G-sets.
Get a map

$$A(G) \longrightarrow [BG, QS^0]$$

Burnside ring of finite G-sets.

Call this the weak Segal conjecture.

Segal conjectured that $A(G)^{\wedge}_{I} \simeq [BG, QS^0]$ for G finite,
I the augmentation ideal, kernel of the map counting points.

Henceforth, everything is stable.

$$A(G) \cong \pi_0^G(\mathbb{S}), \text{ equivariant } \pi_0.$$

$$[BG, QS^0] \cong \pi_0^G(F(EG_+, \mathbb{S})) \cong \pi_0(\mathbb{S}^{hG}).$$

<u>Segal conjecture rephrased</u>: $\mathbb{S}^G \longrightarrow \mathbb{S}^{hG}$ equivalence after completing at the augmentation ideal.

<u>Ex</u>. $G = C_p$, this is equivalent to showing it is an $\simeq$ after p-completion.

Rem. $\mathbb{S}^{C_p} \simeq \mathbb{S} \vee BC_p$ Tom-Dieck splitting.

$\mathbb{S}^{hC_p} \simeq DBC_p$.

This conjecture is known.

C_2	Lin.
C_p	Gunawardena.
C_{p^n}	Ravenel.
$(C_p)^{\times n}$	Adams-Gunawardena-Miller.
Game Over	Carlsson.

II. Motivic homotopy theory.

S base (probably noetherian) scheme.

G, order invertible in S.

$$\begin{tikzcd} Sm_S^G & P(Sm_S^G)_\bullet \arrow[r, "①"] & Spc_\bullet^G(S) \arrow[d, "② \text{ stabilize}"] \\ & & Spt^G(S) \end{tikzcd}$$

① A motivic G-space is a presheaf which is $\mathbb{A}^1$-invariant and Nis-excision.

$\mathcal{F}(X \times \mathbb{A}^1) \simeq \mathcal{F}(X)$

Nisnevich excision...

② V a G-vector bundle on S.
Motivic representation sphere $T^V = V/V \setminus S$.
Now, stabilize w/respect to all of these.
Enough to stabilize w.r.t. the regular rep. sphere T^ρ.

$X \in Sp^G(S)$

$$X^{hG} = \lim_G X \simeq F(E_\cdot G_+, X)^G$$

Ordinary motivic spectra.

$$E_\cdot G = (\cdots G\times G\times G \rightrightarrows G\times G \rightrightarrows G)$$

More intrinsically

$$\mathbb{E}G(W) := \begin{cases} * & W \text{ has free action,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Thm. $\mathbb{E}G/G \simeq BG$.

Geometric classifying space, Morel-Voevodsky, Totaro.

"Improved" homotopy fixed points

$$F(\mathbb{E}G_+, X)^G.$$

Get a sequence

$$X^G \to F(\mathbb{E}G_+, X)^G \to X^{hG}.$$

Conjecture. (Motivic Segal).

$$\pi_{*,*} \mathbb{S}^G \longrightarrow \pi_{*,*} F(\mathbb{E}G_+, \mathbb{S})^G$$

an iso. after suitable completion.

Thm (GHKRØ). Let $S = \mathrm{spec}(k)$, k a field, char $k \neq 2$, $\# |k^\times / k^{\times 2}| < \infty$. Then,

$$\hat{\pi}_{*,*}(\mathbb{S}^{C_2}) \xrightarrow{\simeq} \hat{\pi}_{*,*} F(\mathbb{E}C_{2+}, \mathbb{S})^{C_2}.$$

Completion w.r.t. $2, \eta$.

Note. $\mathbb{S}^{C_2} \simeq \mathbb{S} \vee \mathbb{B}C_2$

$F(\mathbb{E}C_{2+}, \mathbb{S}) \simeq D(\mathbb{B}C_{2,+})$

Motivic Tom-Dieck splitting. Gepner-Heller.

Outline of proof.

1) Motivic Tate diagram.

$$\begin{tikzcd} \mathbb{E}C_{2+} \wedge_{C_2} X \arrow[r] \arrow[d, "\wr"'] & X^{C_2} \arrow[r] \arrow[d] & (\tilde{\mathbb{E}}C_2 \wedge X)^{C_2} \arrow[d] \\ \mathbb{E}C_{2+} \wedge_{C_2} X \arrow[r] & F(\mathbb{E}C_{2+}, X)^{C_2} \arrow[r] & \left[\tilde{\mathbb{E}}C_2 \wedge F(\mathbb{E}C_{2+}, X)\right]^{C_2} \end{tikzcd}$$

$\tilde{\mathbb{E}}C_2 = \mathrm{cofib}(\mathbb{E}C_{2+} \to S^0)$

$\mathbb{E}C_2 = \operatorname{colim}_n \mathbb{A}(n\sigma) \setminus \{0\} \implies \tilde{\mathbb{E}}C_2 \simeq T^{\infty\sigma}$.

sign reps

$\mathbb{A}(n\sigma) = \mathbb{A}^n$, $x \longmapsto -x$

Thm (Gepner-Heller).

1) $\Phi^{C_2}(\Sigma^\infty Y) \simeq \Sigma^\infty Y^{C_2}$.

2) $(\mathbb{E}C_{2+} \wedge X)^{C_2} \simeq \mathbb{E}C_{2+} \wedge_{C_2} X$.

Plug in $X = \mathbb{S}$. Find a pullback square

$$\begin{tikzcd} \mathbb{S}^{C_2} \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \\ F(\mathbb{E}C_{2+}, \mathbb{S}) \arrow[r] & \lim_n \Sigma L^\infty_{-n} \end{tikzcd}$$

where $L^\infty_{-n} = \mathbb{E}C_{2+} \wedge_{C_2} T^{-n\sigma}$,

$L^\infty = \mathbb{B}C_{2+}$.

So, to prove the theorem, can study $\mathbb{S} \longrightarrow \lim_n \Sigma L^\infty_{-n}$.

2) Inverse limit of Adams s.s. for each n.

$$\mathrm{Ext}_{\mathcal{A}}\left(H^{*,*}_{mot}(\Sigma L^{\wedge}_{-n}),\ \mathbb{M}_2\right) \Rightarrow \hat{\pi}_{*,*}(\Sigma L^{\wedge}_{-n})$$

$\uparrow$

motivic Steenrod algebra

Inverse limit of these gives

$$\mathrm{Ext}_{\mathcal{A}}\left(H^{*,*}_{cont}(\Sigma L^{\wedge}_{-\infty}),\ \mathbb{M}_2\right) \Rightarrow \hat{\pi}_{*,*}(\Sigma L^{\wedge}_{-\infty}).$$

3) Motivic Singer construction

$\mathbb{M}_2 \simeq H^{*,*}_{Mot}(pt).$

$$R_+ : \mathcal{A}\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$$

$$R_+(M) \longrightarrow M \quad \text{Tor-equivalence.}$$

4) $H^{*,*}_{cont}(L^{\wedge}_{-\infty}) \cong H^{*,*}(\mathbb{B}C_{2+})[y^{-1}]$,

y Euler class of sign rep.

Also, this gives $R_+(\mathbb{M}_2) \simeq \Sigma H^{*,*}_{cont}(L^{\wedge}_{-\infty}).$

5) Conclude.

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2) & \Longrightarrow & \hat{\pi}_{*,*}(\mathbb{S}) \\ \Big\| \simeq \text{ from Motivic Singer construction.} & & \Big| \\ \mathrm{Ext}_{\mathcal{A}}(R_+(\mathbb{M}_2), \mathbb{M}_2) & \Longrightarrow & \hat{\pi}_{*,*}(\Sigma L^{\wedge}_{-\infty}) \end{array}$$